

# The Horizontal Symmetry for Neutrino Mixing

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## Abstract

We argue that the best way to determine horizontal symmetry is from neutrino mixing, and proceed to show that the only finite group capable of yielding the tri-bimaximal mixing for all Yukawa couplings is  $S_4$ , or any group containing it. The method used is largely group theoretical, but it can be implemented by dynamical schemes in which the Higgs expectation values introduced to break  $S_4$  spontaneously are uniquely determined up to an unknown scale for each.

## I. INTRODUCTION

Progress in particle physics is often guided by symmetry. From isospin to the eightfold way, from the Standard Model (SM) to GUT, SUSY and superstrings, symmetry always plays a central role. It is therefore natural to expect that symmetry may open the door to the generation problem as well. For that reason, a plethora of horizontal symmetry groups have been proposed, including  $Z_m$ ,  $Z_m \times Z_n$ ,  $D_n$ ,  $S_3$ ,  $S_4$ ,  $A_4$ ,  $A_5$ ,  $T'$ ,  $\Delta(27)$ ,  $SO(3)$ ,  $SU(3)$ , and others. The reason why so many diverse groups can all claim to be reasonable candidates is the presence of many adjustable Yukawa coupling constants and Higgs expectation values in these models. By suitably tuning these parameters one can arrive at many attractive results.

If there is indeed a horizontal symmetry in nature, it must be unique, and we need a criterion to determine what it is. I subscribe to the view that a true symmetry would reveal itself without any tuning of the dynamical parameters, and I shall use that as the criterion to determine the horizontal symmetry. I argue that neutrino mixing, rather than quark mixing or the fermion masses, is the proper vehicle to fix such a symmetry. This latter assertion may be contrary to the instinct built up from atomic physics, where approximate symmetry is reflected by proximity of energy levels. In particle physics, symmetries are often broken spontaneously by a large extent, rendering mass spectra useless for recovering the unbroken symmetry. For example, in SM, the bottom and the top quarks belong to the same isodoublet, but their masses are so vastly different that no trace is left of the isospin symmetry. Similarly, the masses of the quarks and charged leptons in different generations are also very different, suggesting that horizontal symmetry is also broken spontaneously and fermion masses are useless in its recovery. I also think that quark mixing, being small, may result from a complicated dynamical perturbation of the unmixed state, whereas neutrino mixing, being large and regular, can best be used to find out the unperturbed and the unbroken horizontal symmetry. The regularity of tri-bimaximal mixing [1] of neutrinos is analogous to the regularity of the Balmer series for hydrogen atom. The latter led to the discovery of the Bohr atom, with its rotational and dynamical symmetry of a Coulomb potential, but it cannot predict fine structures and hyperfine structures of the spectra brought on by additional dynamical perturbations. Similarly, the tri-bimaximal mixing may also be subject to a small perturbation which future experiments will reveal, but that does not invalidate the

horizontal symmetry established by its use.

I shall show in this letter and a subsequent detailed paper [2] that  $S_4$ , the permutation group of four objects and the symmetry group of the octahedron and the cube, is the only finite group capable of giving rise to tri-bimaximal mixing without tuning parameters. This symmetry is *unique* up to the obvious generalization, that any group containing  $S_4$  is a possible horizontal group as well. To avoid repetition, when we say  $S_4$  is unique from now on, we always mean to include this possible extension.

Since we like to uncover the symmetry without resorting to specific dynamics, the method employed is largely group theoretical, but we will discuss the implementation of some dynamical schemes at the end. In that case,  $S_4$  is broken by the introduction of Yukawa couplings and non-SM Higgs bosons. The Higgs expectation values are uniquely determined by the group structure, up to unknown scales that will be absorbed into the Yukawa coupling constants to form ‘effective coupling constants’, to be used to fit the leptonic masses. Since there are now additional Higgs present to share the burden of fermion masses, the coupling of the SM Higgs to leptons are *no longer* proportional to their masses.

Much has been written about the  $S_4$  subgroup  $A_4$  as a horizontal group [3]. However,  $A_4$  gives rise naturally only to trimaximal mixing but not bimaximal mixing [4]. It requires either a tuning of the Yukawa couplings [5], or the additional symmetries contained in  $S_4$  to get the bimaximal mixing. The group  $S_4$  had been previously studied [6], but with a different motivation and a different conclusion.

## II. FROM TRI-BIMAXIMAL MIXING TO $S_4$

After reviewing [4] how  $S_4$  comes about, the argument for its uniqueness will be outlined.

Let  $c = (e_L, \mu_L, \tau_L)^T$  be the left-handed charged leptons and  $\nu = (\nu_e, \nu_\mu, \nu_\tau)^T$  the left-handed Majorana neutrinos. Instead of their mass matrices  $M_c$  and  $M_\nu$ , we study the combination  $\bar{M}_c = \sqrt{M_c M_c^\dagger}$  and  $M_\nu$ , because they connect left-handed to left-handed fermions, thereby avoiding the involvement of the right-handed fermions in this symmetry analysis.  $\bar{M}_c$  is hermitean and  $M_\nu$  symmetric; they can be diagonalized by unitary matrices  $U_c$  and  $U_\nu$ , so that  $U_c^\dagger \bar{M}_c U_c$  is the diagonal matrix of charge-lepton masses, and  $U_\nu^T M_\nu U_\nu$  is the diagonal matrix of neutrino masses. The PMNS mixing matrix is given by  $U = U_c^\dagger U_\nu$ . If  $F$  is a symmetry operation of  $c$  and  $G$  a symmetry operation of  $\nu$ , both unitary, then

under the transformations  $c \rightarrow Fc$  and  $\nu \rightarrow G\nu$ , symmetry demands  $F^\dagger \bar{M}_c F = \bar{M}_c$  and  $G^T M_\nu G = M_\nu$ . As shown in [4], this means that the eigenvectors of  $F$  are the columns of  $U_c$ , with eigenvalues of unit modulus, and the eigenvectors of  $G$  are the columns of  $U_\nu$ , with eigenvalues  $\pm 1$ . We shall choose the sign of  $G$  so that it has one  $+1$  eigenvalue and two  $-1$ 's..

It follows [4] that if  $F = G$ , then  $U_c = U_\nu$  and  $U = \mathbf{1}$ . This is false, hence the horizontal symmetry must be broken to enable  $F \neq G$ , and we assume the breaking to be spontaneous. In the basis where  $M_c$  is diagonal, which we adopt from now on,  $F$  is diagonal and  $U = U_\nu$ . Hence the neutrino symmetry operator  $G$  can be read off from the tri-bimaximal mixing matrix  $U$ . There are three of them, with the eigenvector of  $G_i$  ( $i = 1, 2, 3$ ) of eigenvalue  $+1$  taken from the  $i$ th column of  $U$ , and the other two eigenvectors of eigenvalues  $-1$  taken from the other two columns. See [4] for details and formulas. These three matrices commute, with  $G_1 = G_2 G_3$ , so the group containing  $G_2$  and  $G_3$  must also automatically contain  $G_1$ . The minimal horizontal group appropriate to tri-bimaximal mixing is therefore the finite group  $\mathcal{G} = \{F, G_2, G_3\}$  generated by  $F, G_2$ , and  $G_3$ . This group is not a priori unique because  $F$  is not. However, since  $\mathcal{G}$  is assumed to be finite, there must be an integer  $n$  such that  $F^n = \mathbf{1}$ . Conversely, given a finite group  $\mathcal{G}$ , it can be spontaneously broken to reveal the tri-bimaximal mixing without tuning only when three of its members,  $F, G_2, G_3$ , can be found to have these properties when  $\bar{M}_c$  is diagonal. Since  $\bar{M}_c$  is not known from  $\mathcal{G}$ , the only way to guarantee its diagonality is to go to the basis where  $F$  is diagonal. Since  $F$  commutes with  $\bar{M}_c$ , the diagonality of  $\bar{M}_c$  is guaranteed if the three eigenvalues of  $F$  are different, so we shall demand that of  $F$  from now on. In particular, this requires  $n \geq 3$ . For  $n = 3$ , the three entries of  $F$  must be  $1, \phi = \exp(2\pi i/3)$ , and  $\phi^2$ . There are  $3! = 6$  possible  $F$ 's obtained from different positioning of these three eigenvalues, but they only generate two different groups,  $\mathcal{G} = S_4$ , and  $3.S_4$  [7]. The latter is obtained by adjoining  $S_4$  with  $\phi S_4$  and  $\phi^2 S_4$ , and it contains  $S_4$  as a subgroup. So for  $n = 3$ , the minimum horizontal group is  $S_4$ .

To prove the uniqueness of  $S_4$ , we must show that no other finite group (except those containing  $S_4$ ) can be so generated for  $n > 3$ . A direct proof is difficult because there are an infinite number of cases to consider, so we shall resort to a different strategy. Since an overall scalar factor multiplying a matrix does not alter its eigenvectors, which are all that we care in order to get the correct  $U$ , we may confine ourselves to finite subgroups of  $SU(3)$

and  $SO(3)$ , or their central extensions. We must show that unless the finite group contains  $S_4$ , otherwise it is impossible to find three members  $F, G_2, G_3$  in it so that in the basis where  $F$  is diagonal, the invariant eigenvectors of  $G_2$  and  $G_3$  are given by the second and third columns of the tri-bimaximal matrix  $U$ . This strategy is more viable than a direct approach because all the finite subgroups of  $SO(3)$  (or  $SU(2)$ ) and  $SU(3)$  are known.

For  $SO(3)$  (or  $SU(2)$ ) [8], they are given by the two infinite series,  $Z_n$  (cyclic groups) and  $D_n$  (dihedral groups), and three isolated ones:  $A_4$ , the alternating group of 4 objects, which is also the symmetry group of the tetrahedron;  $S_4$ , the symmetric group of four objects, which is also the symmetry group of the octahedron and the cube; and  $A_5$ , the symmetry group of the icosahedron and the dodecahedron. For  $SU(3)$  [8, 9], there are again two infinite series,  $\Delta(3n^2)$  and  $\Delta(6n^2)$ , and six isolated ones,  $\Sigma(36), \Sigma(60), \Sigma(72), \Sigma(168), \Sigma(216)$ , and  $\Sigma(360)$ ; the number in each case indicates the order of the group. The detailed argument to reject all of them except  $S_4$  is somewhat lengthy, and will be postponed to another publication [2]. However, it is easy to state on what basis each of them is rejected. First of all, it has been shown in [4] that the group must possess a three-dimensional irreducible representation, or else we cannot get the tri-bimaximal mixing pattern without tuning parameters. On that basis the groups  $Z_n, D_n, \Sigma(36), \Sigma(72), \Sigma(360)$  can be rejected because they do not possess three-dimensional irreducible representations. As mentioned before, the group  $A_4$  is rejected because it leads to trimaximal but not bimaximal mixing [4]. The groups  $\Delta(3n^2)$  and  $\Delta(6n^2)$  are rejected because their explicitly-known three-dimensional irreducible representations all have a special form, so special that tri-bimaximal mixing cannot occur unless they contain  $S_4$  as a subgroup. The rest of the groups are rejected by using their character tables to pick out the order  $n$  of  $F$  and its eigenvalues. If  $n = 3$ , then either the group contains  $S_4$  or else it cannot accommodate the tri-bimaximal mixing. For  $n > 3$ , we can use its eigenvalues to construct all possible  $F$ . With  $G_2, G_3$  determined from the columns of the tri-bimaximal matrix, we can compute the orders of  $FG_2$  and  $FG_3$ , and in each case one or the other would have an order larger than the order of the whole finite group. Hence at least one of these  $G_2$  and  $G_3$  cannot be in the group and tri-bimaximal mixing cannot occur.

### III. SPONTANEOUS BREAKING

The discussion so far is purely group theoretical. To implement a dynamical scheme complementary to the discussion we have to write down the mass term of an effective Hamiltonian. After integrating over the right-handed fermions, it can be symbolically written as

$$H = \sum_A \left( \lambda_A c^\dagger c \phi^A + \mu_A \nu^T \nu \psi^A \right) + h.c., \quad (1)$$

where  $\lambda_A$  and  $\mu_A$  are the Yukawa coupling constants to the Higgs fields  $\phi^A$  and  $\psi^A$ . For later convenience, an energy scale is incorporated into the couplings so that the Higgs fields become dimensionless, with vacuum expectation values given later. The Higgs fields in (1) may be composite, and the spacetime structure is implicit, may even be non-local, but all that we care is the  $S_4$  behavior. Before the Higgs bosons develop an expectation value,  $H$  must be invariant under every  $S_4$  transformation. Afterwards, the horizontal symmetry is broken,  $\langle H \rangle$  is no longer invariant under every  $S_4$ , but it must still be invariant under the residual symmetries of  $F$  on  $c$ , and  $G_2, G_3$  on  $\nu$ , in order to recover the tri-bimaximal mixing. To achieve that, we must have

$$F \langle \phi^A \rangle = \langle \phi^A \rangle, \quad G_{2,3} \langle \psi^A \rangle = \langle \psi^A \rangle. \quad (2)$$

These equations determine the structure of the vacuum expectation value for every Higgs boson up to an unknown scale which has been incorporated into the Yukawa couplings.

$S_4$  has five irreducible representations,  $\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{3}, \mathbf{3}'$ , and by definition the left-handed fermions belong to  $\mathbf{3}$ . If we use a boldface superscript to denote an irreducible representation, then the representations of  $F$  and  $G_i$  ( $i = 2, 3$ ) are:  $F^{\mathbf{1}} = F^{\mathbf{1}'} = G_i^{\mathbf{1}} = G_i^{\mathbf{1}'} = -G_i^{\mathbf{3}'} = 1$ ,  $F^{\mathbf{2}} = \text{diag}(\emptyset, \emptyset^2)$ ,  $F^{\mathbf{3}} = F^{\mathbf{3}'} = \text{diag}(1, \emptyset, \emptyset^2)$ ,  $G_2^{\mathbf{2}} = \text{diag}(1, 1)$ , and

$$G_3^{\mathbf{2}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_2^{\mathbf{3}} = G_2^{\mathbf{3}'} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad G_3^{\mathbf{3}} = -G_3^{\mathbf{3}'} = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3)$$

Applying this to (2), we deduce that  $\langle \phi^{\mathbf{1}} \rangle = \langle \phi^{\mathbf{1}'} \rangle = \langle \psi^{\mathbf{1}} \rangle = 1$ ,  $\langle \psi^{\mathbf{1}'} \rangle = \langle \psi^{\mathbf{3}} \rangle = \langle \phi^{\mathbf{2}} \rangle = 0$ ,  $\langle \phi^{\mathbf{3}} \rangle = \langle \phi^{\mathbf{3}'} \rangle = (1, 0, 0)^T$ ,  $\langle \psi^{\mathbf{2}} \rangle = (1, 1)^T$ , and  $\langle \psi^{\mathbf{3}'} \rangle = (1, 1, 1)^T$ . Since  $\mathbf{3} \times \mathbf{3}$  produces  $\mathbf{1} + \mathbf{2} + \mathbf{3} + \mathbf{3}'$ , neither  $\phi^{\mathbf{1}'}$  nor  $\psi^{\mathbf{1}'}$  is present in (1). With  $\langle \phi^{\mathbf{2}} \rangle = \langle \psi^{\mathbf{3}} \rangle = 0$ , there remain exactly three Yukawa coupling constants each for the charged leptons and neutrinos in  $\langle H \rangle$ ,

just enough to fit the three charged lepton masses and the three neutrino masses. With appropriate Clebsch-Gordan coefficients inserted, the mass matrices can be read off from (1) to be  $\bar{M}_c = \text{diag}(a - 2b, a + b - c, a + b + c)$ , where  $a = l^1/\sqrt{3}$ ,  $b = l^{3'}/\sqrt{6}$ ,  $c = l^3/\sqrt{2}$ , and

$$M_\nu = \begin{pmatrix} c - 2e & d + e & d + e \\ d + e & d - 2e & c + e \\ d + e & c + e & d - 2e \end{pmatrix}, \quad (4)$$

where  $d = \mu^1/\sqrt{3}$ ,  $d = \mu^2/\sqrt{3}$ ,  $e = \mu^{3'}/\sqrt{6}$ . Since  $M_\nu$  is 2-3 symmetric and magic, the tri-bimaximal mixing pattern is guaranteed [10].

So far we have ignored the right-handed leptons. They must be introduced to implement a local dynamics, but there is more than one way to do so. For example, if the right-handed charged leptons, denoted by  $c_R$ , belongs to **3**, then the Hamiltonian is once again given by (1), with  $c^\dagger$  replaced by  $c_R^\dagger$ . The subsequent  $S_4$  analyzes are identical, so we have three isodoublet Higgs coupled to the charged leptons, in representations **1, 3, 3'**, with the  $S_4$ -singlet identified with the SM Higgs, and in addition, three isotriplet Higgs in **1, 2, 3'** coupled to the Majorana neutrinos. Other dynamical schemes and the allowed Yukawa potentials will be discussed in a separate publication later.

In conclusion, we have shown that  $S_4$ , and groups containing it, are the only finite horizontal symmetries capable of reproducing tri-bimaximal mixing of neutrinos without tuning the Yukawa coupling constants. These constants are used exclusively to fit the fermionic masses.

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